p. 481-486 (NASA RP-82)

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Reprinted for private circulation from
THE ASTROPHYSICAL JOURNAL
Vol. 138, No. 2, August 15, 1963

N64 11197

CODE NONE

TRANSFER OF ANGULAR MOMENTUM BETWEEN EJECTED PARTICLES AND THE BINARY SYSTEM

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Repr. from Astrophys. J., v. 138, no. 2, Aug. 15, 1963

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Received February 11, 1963; revised March 28, 1963

Uncl.

11197

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ABSTRACT

Transfer of angular momentum between the ejected particle and the binary system itself has been studied numerically in the framework of the restricted three-body problem and the general behavior of such transfer for particles of high velocities of ejection is elucidated. Tables are given for the angular momentum that will be carried away to infinity under various initial conditions of ejection.

AUTHOR refs

In the previous paper (Huang 1963) we have seen that the change in orbital period of a close binary system as a result of mass ejection by its components depends greatly on the angular momentum per unit mass, h_e, of the escaped particles. Therefore, it is important to know how much angular momentum is carried away per unit mass under any given initial condition. It is the purpose of this note to give, by numerical computation, some ideas about the behavior of angular-momentum transfer between the ejected particles and the binary system itself and to show that h_e of a particle at infinity, denoted by $h_{e,\infty}$, may be obtained for a given set of initial conditions by integrating the equations of motion of the particle over a relatively short period of time. All symbols used here have the same meanings as in the previous paper unless otherwise stated.

Since we are interested only in close binaries, it is permissible to set $e = 0$. Then the equations of motion become identical with those in the restricted three-body problem. In this paper we shall investigate the transfer of angular momentum by the use of these equations.

Several authors, including Kuiper (1941), Kopal (1956, 1957), and Mrs. Gould (1957, 1959), have derived many orbits for gaseous particles in a close binary system from the solutions of the restricted three-body problem. One disadvantage of this kind of calculation is that it is impossible to compute the orbits and plot them for all conceivable initial conditions. Also, even if they were all plotted, it is difficult, if not impossible, to derive from these highly complicated and seemingly irregular orbits much information that may throw some light on the actual motion of gaseous particles in the system. Worst of all, such an approach necessarily neglects the collisions of particles themselves, although collisions cannot be neglected in this case (Prendergast 1960). Consequently, with only a few exceptions, direct integration of the equations of motion has yielded few results of any great physical significance. However, as we shall see, useful information can be obtained by investigating the angular momentum of the particle as a function of time.

Let us choose a co-ordinate system (x, y) rotating with the circular motion of the binary stars around the center of mass of the system. Furthermore, we take the total mass of the system as the unit of mass, the separation between the two components as the unit of length, and $1/(2\pi)$ of the orbital period as the unit of time. Thus, if μ denotes the mass of one component that is located at $(1 - \mu, 0)$, the other component will have mass $1 - \mu$ and be located at $(-\mu, 0)$ in the rotating co-ordinate system.

The equations of motion can then be written in a dimensionless form (e.g., Moulton 1914), which will not be given here. The angular momentum per unit mass in the present system of units, h_e, is now given by

$$h_e = x\frac{dy}{dt} - y\frac{dx}{dt} + x^2 + y^2 . \tag{1}$$

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From the equations of motion, we can easily show that

$$h_e = \int_{t_0}^{t} \frac{x_1 x_2 y}{r^3} \left\{ \left[1 - \left(\frac{2 x x_2}{r^2} - \frac{x_2^2}{r^2} \right) \right]^{-3/2} - \left[1 - \left(\frac{2 x x_1}{r^2} - \frac{x_1^2}{r^2} \right) \right]^{-3/2} \right\} dt , \tag{2}$$

where $r^2 = x^2 + y^2$ and $x_1 = -\mu$ and $x_2 = 1 - \mu$.

When the third body is far away from the system, the changes in x and in y are dominantly due to rotation of the co-ordinate system. Therefore, we may write as a first approximation,

$$x = r \cos t , \qquad y = -r \sin t . \tag{3}$$

If we substitute x and y given by equations (3) in equation (2), we find that h_e assumes the following form:

$$\int_{t_0}^{t} \sin t dt f (r, \cos t) .$$

If we regard r as constant, the integral vanishes when we integrate over a complete period of the binary motion, i.e., from $t = t_0$ to $t = t_0 + 2\pi$. Thus we derive the conclusion that no net angular momentum is transferred in a physically significant degree over a period when the particle is far away from the system.

If we expand the expressions in the square brackets in the integrand of the integral in equation (2) in terms of $1/r$, and take only the first term, we obtain

$$h_e = \frac{3 \mu (1 - \mu)}{4 r^3} (\cos 2 t_0 - \cos 2 t) , \tag{4}$$

if we again make use of equations (3) and set r as constant during integration. Thus the angular momentum of the particle at any moment follows a double sinusoidal curve of decreasing amplitude as r increases. The curve has maxima at $x = 0$ and minima at $y = 0$. Results of actual computations completely verify this prediction, which is based on our approximate calculation. In Figure 1 we have illustrated for $t > 7.2$ the variation of angular momentum of the third body ejected from one of the components under the initial conditions

$$x = 0.85 , \qquad y = 0 , \qquad \frac{dx}{dt} = 4 , \qquad \text{and} \qquad \frac{dy}{dt} = 0 ,$$

with $\mu = 0.3$. This kind of fluctuation in angular momentum is common to all particles that are escaping to infinity with reasonable speeds. Since the angular momentum which a particle under a given set of initial conditions will possess at infinity can be calculated, we now have only one value instead of an endless orbit to associate with a given set of initial conditions.

Another advantage of treating the angular momentum instead of the orbit itself comes from the consideration of collisions. In the case of the orbital method, a collision between two particles transforms the orbits to two completely different orbits. We can predict nothing about the behavior of the orbits before and after the collision. It is for this reason that the orbit approach faces its greatest weakness in treating the motion of gas in close binaries.

The situation becomes quite different when we use the angular-momentum approach because a definite relation exists that links the dynamical state of particles before and after collision. The total angular momentum of colliding particles is conserved in the process of collision. If many particles are involved in collision, the net result of interchange in angular momentum among the colliding particles tends to equalize the angular momenta of individual particles. Thus the chaotic motion of particles will turn more or less into streams. This is why gaseous rings can be formed by ejected matter. The angular momentum per unit mass in the stream can be obtained by taking the average of the same quantity for all the individual particles involved. In this way we may look upon the angular-momentum consideration as a link between the orbital approach and the hydrodynamic approach to the problem of stream motion in the binary system. While the present paper does not include a study of collision, it is evident that, for fast ejection, the results tabulated here can be applied, as an approximation, to aggregates of particles if we use the average values as the initial conditions.

We have computed the stabilized angular momentum for eight groups of sets of initial conditions that are given in Table 1, where r_1 and r_2 represent, respectively, the radii of the two components. These groups include the cases of ejection in four mutual perpendicular directions for each component, as the integration has been confined in the $x - y$ plane. The computation was carried out on the 7090 IBM computer at the Goddard Space Flight Center, the machine program being written and actual computation super-

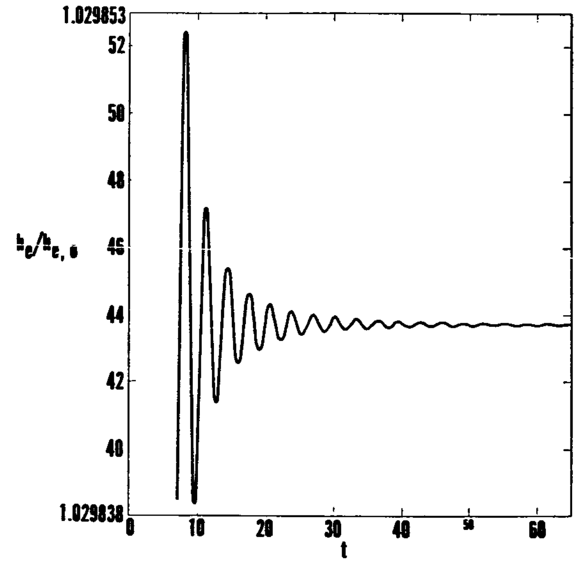

FIG. 1.—The variation in angular momentum of a particle escaping from a binary system. It is the typical behavior of an escaping particle that its angular momentum undergoes damped oscillations before reaching a value that it will carry to infinity.

TABLE 1

EIGHT GROUPS OF INITIAL CONDITIONS

No.	x	y	dx/dt	dy/dt	Type of Ejection
1.	$1-\mu$	r_2	0	V	Front
2.	$1-\mu$	$-r_2$	0	V	Rear
3.	$1-\mu+r_2$	0	V	0	Exterior
4.	$1-\mu-r_2$	0	V	0	Interior
5.	$-\mu$	$-r_1$	0	V	Front
6.	$-\mu$	r_1	0	V	Rear
7.	$-\mu-r_1$	0	V	0	Exterior
8.	$-\mu+r_1$	0	V	0	Interior

vised by Clarence Wade, Jr. An accuracy of at least six significant figures was maintained throughout, as judged by the constancy of the Jacobian constant. In most cases an accuracy of eight significant figures has been achieved over the entire range of integration.

As may be expected intuitively, the transfer of angular momentum from the binary system to the particle takes place most appreciably when the particle is near to one of the components, and this is verified by our extensive computations. Therefore, one cannot easily predict the angular momentum of a particle as a function of the time when it is confined to the system.

Our interest, however, concerns only the escaped particles, for which the pattern of transfer is relatively simple. In the case of ejection from the front of the star (i.e., in the direction of its orbital motion), the angular momentum of the ejected particle starts from a positive value, decreases gradually, and then stabilizes to a smaller positive value after a series of fluctuations mentioned before and shown in Figure 1. In the case of ejection from the rear side of the star (i.e., against the direction of its orbital motion), its angular momentum is negative at the time of ejection. It increases through the interaction with the binary system itself and, as in the other case, stabilizes through fluctuations to a negative value. Thus, in both cases, the continuous interaction after ejection neutralizes a part of the angular momentum acquired at the time of ejection.

In the case of outward ejection along the x-axis (positive for the component with mass μ and negative for the other component), the angular momentum initially has a positive value, increases gradually, and then stabilizes, after damped oscillations, to a constant value without further significant change afterward. Thus a particle finally possesses more angular momentum than its initial value. In all cases the faster the initial velocity of ejection, the shorter the time for the particle to reach the stabilized value of angular momentum. Also, the angular momentum is stabilized to a value nearer to its initial value in the case of a fast ejection than in the case of slower ejections.

The case of inner ejection along the x-axis (i.e., negative velocity for the μ component and positive velocity for the $1 - \mu$ component) is more complicated than the other cases, as we would expect. Actual computations show that it differs from the other cases mainly when the velocity of ejection is high, for then the particle will either simply fall into the companion component or be strongly perturbed by it during a close encounter such that the angular momentum is greatly modified. Thus we cannot make a general statement about the angular momentum for high velocities of ejection. However, it is interesting to note that for intermediate velocities, say between 4 and 15 in our units, the stabilized value of angular momentum does not vary greatly with the initial velocity.

Some of the numerical results obtained for various sets of initial conditions are given in Table 2 according to the order listed in Table 1. For the first four groups we have adopted $r_2 = 0.15$, while for the last four groups $r_1 = 0.20$. Five values for μ and four values for V have been used for each group. There are two entries for each combination of μ and V, the first one being the initial angular momentum $h_{e,0}$ corresponding to the initial conditions and the second being the stabilized value of angular momentum, i.e., $h_{e,\infty}$. The values $h_{e,\infty}$ in most cases were obtained after we integrated the equations of motion up to $t = 20$. But in some cases where the convergence is slow, we have to reach $t = 50$ before a stabilized value can be determined accurately.

Only velocities greater than 4 are included in the table because it is difficult to present a clear-cut picture of the ejected particle even when the result is represented in terms of angular momentum. In other words, the present analysis does not help much the problem of gaseous motion inside the close binary system, but, when combined with the results obtained in the previous paper, it does help us understand the effect of ejection of mass to infinity on the orbital period of the binary. For higher velocities of ejection than the listed values, the particle will practically carry the initial value of angular momentum to infinity, except in the case of interior ejection.

TABLE 2

ANGULAR MOMENTUM OF AN ESCAPED PARTICLE AS A FUNCTION OF INITIAL CONDITIONS

No. of Group	V		$\mu=0.1$	$\mu=0.2$	$\mu=0.3$	$\mu=0.4$	$\mu=0.5$
1 ($r_2=0.15$)	4	$h_{e,0}$	4.4325	3.8625	3.3125	2.7825	2.2725
		$h_{e,\infty}$	4.299	3.620	2.987	2.402	1.8669
	6	$h_{e,0}$	6.2325	5.4625	4.7125	3.9825	3.2725
		$h_{e,\infty}$	6.145	5.305	4.503	3.742	3.019
	10	$h_{e,0}$	9.8325	8.6625	7.5125	6.3825	5.2725
		$h_{e,\infty}$	9.780	8.569	7.390	6.242	5.125
	15	$h_{e,0}$	14.3325	12.6625	11.0125	9.3825	7.7725
		$h_{e,\infty}$	14.298	12.600	10.931	9.290	7.676
2 ($r_2=0.15$)	−4	$h_{e,0}$	−2.7675	−2.5375	−2.2875	−2.0175	−1.7275
		$h_{e,\infty}$	−2.647	−2.320	−1.9970	−1.6802	−1.3703
	−6	$h_{e,0}$	−4.5675	−4.1375	−3.6875	−3.2175	−2.7275
		$h_{e,\infty}$	−4.486	−3.991	−3.494	−2.995	−2.493
	−10	$h_{e,0}$	−8.1675	−7.3375	−6.4875	−5.6175	−4.7275
		$h_{e,\infty}$	−8.118	−7.249	−6.371	−5.484	−4.588
	−15	$h_{e,0}$	−12.6675	−11.3375	−9.9875	−8.6175	−7.2275
		$h_{e,\infty}$	−12.635	−11.277	−9.910	−8.528	−7.134
3 ($r_2=0.15$)	4	$h_{e,0}$	1.1025	0.9025	0.7225	0.5625	0.4225
		$h_{e,\infty}$	1.1107	0.9178	0.7440	0.5890	0.4523
	6	$h_{e,0}$	1.1025	0.9025	0.7225	0.5625	0.4225
		$h_{e,\infty}$	1.1065	0.9097	0.7322	0.5739	0.4348
	10	$h_{e,0}$	1.1025	0.9025	0.7225	0.5625	0.4225
		$h_{e,\infty}$	1.1040	0.9053	0.7262	0.5667	0.4270
	15	$h_{e,0}$	1.1025	0.9025	0.7225	0.5625	0.4225
		$h_{e,\infty}$	1.1033	0.9038	0.7242	0.5645	0.4245
4 ($r_2=0.15$)	−4	$h_{e,0}$	0.5625	0.4225	0.3025	0.2025	0.1225
		$h_{e,\infty}$	0.8265	0.8954	0.9264	0.9172	0.8659
	−6	$h_{e,0}$	0.5625	0.4225	0.3025	0.2025	0.1225
		$h_{e,\infty}$	0.8201	0.8819	0.9067	0.8936	0.8421
	−10	$h_{e,0}$	0.5625	0.4225	0.3025	0.2025	0.1225
		$h_{e,\infty}$	0.8156	0.8730	0.8941	0.8788	0.8269
	−15	$h_{e,0}$	0.5625	0.4225	0.3025	0.2025	0.1225
		$h_{e,\infty}$	0.8138	0.8693	0.8891	0.8730	0.8209
5 ($r_1=0.2$)	−4	$h_{e,0}$	0.4500	0.8800	1.3300	1.8000	2.2900
		$h_{e,\infty}$	0.3414	0.6907	1.0862	1.5262	2.010
	−6	$h_{e,0}$	0.6500	1.2800	1.9300	2.6000	3.2900
		$h_{e,\infty}$	0.5839	1.1633	1.7779	2.427	3.111
	−10	$h_{e,0}$	1.0500	2.0800	3.1300	4.2000	5.2900
		$h_{e,\infty}$	1.0121	2.013	3.042	4.100	5.186
	−15	$h_{e,0}$	1.5500	3.0800	4.6300	6.2000	7.7900
		$h_{e,\infty}$	1.5252	3.036	4.572	6.134	7.721

TABLE 2—*Continued*

No. of Group	V		$\mu=0.1$	$\mu=0.2$	$\mu=0.3$	$\mu=0.4$	$\mu=0.5$
6 ($r_1=0.2$)	4	$h_{e,0}$	−0.3500	−0.7200	−1.0700	−1.4000	−1.7100
		$h_{e,\infty}$	−0.2598	−0.5613	−0.8639	−1.1669	−1.4697
	6	$h_{e,0}$	−0.5500	−1.1200	−1.6700	−2.2000	−2.7100
		$h_{e,\infty}$	−0.4909	−1.0154	−1.5334	−2.045	−2.549
	10	$h_{e,0}$	−0.9500	−1.9200	−2.8700	−3.8000	−4.7100
		$h_{e,\infty}$	−0.9147	−1.8572	−2.788	−3.706	−4.613
	15	$h_{e,0}$	−1.4500	−2.9200	−4.3700	−5.8000	−7.2100
		$h_{e,\infty}$	−1.4264	−2.878	−4.315	−5.737	−7.144
7 ($r_1=0.2$)	−4	$h_{e,0}$	0.0900	0.1600	0.2500	0.3600	0.4900
		$h_{e,\infty}$	0.1004	0.1773	0.2713	0.3830	0.5126
	−6	$h_{e,0}$	0.0900	0.1600	0.2500	0.3600	0.4900
		$h_{e,\infty}$	0.0940	0.1669	0.2589	0.3699	0.5000
	−10	$h_{e,0}$	0.0900	0.1600	0.2500	0.3600	0.4900
		$h_{e,\infty}$	0.0914	0.1625	0.2532	0.3637	0.4938
	−15	$h_{e,0}$	0.0900	0.1600	0.2500	0.3600	0.4900
		$h_{e,\infty}$	0.0906	0.1611	0.2515	0.3617	0.4918
8 ($r_1=0.2$)	4	$h_{e,0}$	0.0100	0.0000	0.0100	0.0400	0.0900
		$h_{e,\infty}$	0.2968	0.5220	0.7059	0.8428	0.9296
	6	$h_{e,0}$	0.0100	0.0000	0.0100	0.0400	0.0900
		$h_{e,\infty}$	0.2976	0.5148	0.6891	0.8184	0.9020
	10	$h_{e,0}$	0.0100	0.0000	0.0100	0.0400	0.0900
		$h_{e,\infty}$	0.2948	0.5073	0.6768	0.8027	0.8848
	15	$h_{e,0}$	0.0100	0.0000	0.0100	0.0400	0.0900
		$h_{e,\infty}$	0.2933	0.5040	0.6720	0.7968	0.8783

It is a pleasure to express my sincere thanks to Mr. Clarence Wade, Jr., who performed the integration on the IBM 7090 computer at our center and to Mrs. Priscilla Weck, who plotted, in the course of the present investigation, a number of curves, including the one presented here.

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